Some Remarkable Spheres in Connection with Tetrahedra

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Abstract

For triangles we have a well-known theorem, called Miquel's theorem: Choose any point on each side of a triangle and draw through each vertex a circle passing through the two points on the sides which concur in that vertex. Then these three circles always have a point in common, called Miquel's point. For tetrahedra we have an exact analogon of this fact, Roberts' theorem: Choose any point on each edge of a tetrahedron and draw through each vertex a sphere passing through the three points on the edges which concur in that vertex. Then these four spheres always have a point in common. The proof of this fact is quite esthetic and gives rise to the assumption that its method - using stereographic projection and several times the corresponding fact for the triangle - could possibly be applied for similar theorems for tetrahedra.

1. Introduction and Roberts' Theorem

In this paper I would like to present a theorem concerning the general tetrahedron. Although its first (and synthetic) proof was found around 1880 by the british mathematician Samuel Roberts (1827 - 1913), the theorem seems to be almost forgotten (Figure 1):

Given a general tetrahedron choose any point (but no vertex) on each edge and draw through each vertex a sphere passing through the three points on the edges which concur in that vertex. Then these four spheres always have a point in common. ([NAC64]).

2. Miquel's Triangle Theorem

To proof Roberts' theorem we need the analogon in the plane, which is the well-known triangle theorem of Auguste Miquel, established in 1838 (Figure 2):

Given a triangle choose any point (but no vertex) on each side and draw through each vertex a circle passing through the two points on the sides which concur in that vertex. Then these three circles always have a point in common, called Miquel's point.

Proof
Let A, B, C be the triangle vertices and P, Q, R the chosen points on the sides opposite A, B, C respectively. First look at the circles through A and B intersecting in R on side AB and having as second intersection point M. The circle through A has an inscribed quadrangle $ARMQ$, which meens that opposite angles are supplementary, so we have $\angle AQM$ equal to $\angle BRM$. The same is true for the quadrangle $BPMR$ on the circle through B, and so $\angle CPM$ is equal to $\angle BRM$. It follows that $\angle CPM$ is equal to $\angle AQM$, which meens that quadrangle $CPMQ$ is cyclic or M is on the circle through C, thus being the common point of all three circles. We quietly assumed that M is inside the triangle. The other cases of M coinciding with Q or R or lying outside the triangle are treated similarly. □

3. Spherical Version of Miquel's Triangle Theorem

We also need the spherical version of Miquel's theorem (Figure 3):

Given three circles on a sphere having a point D in common and intersecting pairwise in points A, B, C. Choose any point (except

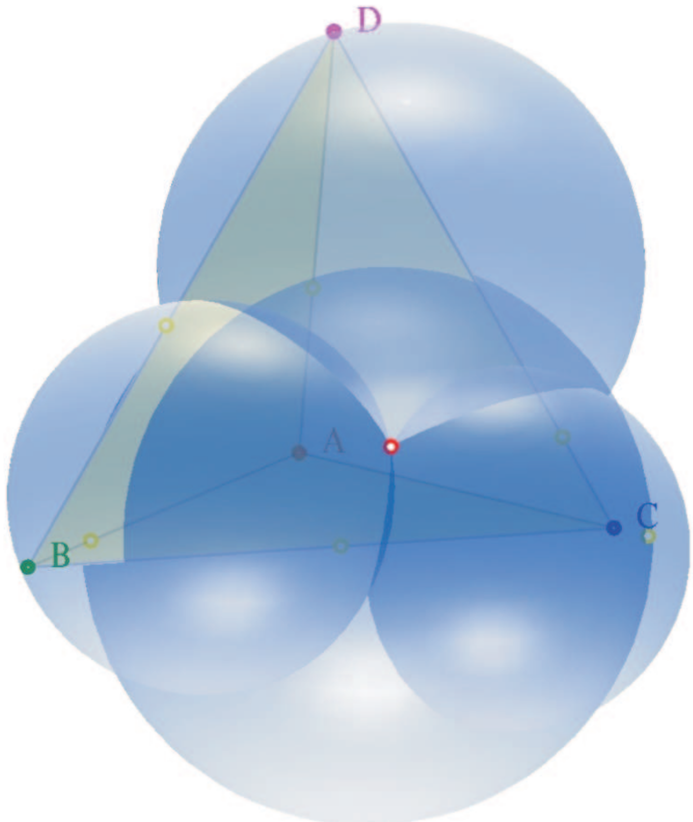

Figure 1: *Roberts' Theorem.*

A,B,C,D) on each circle, P on circle cBC, Q on circle cAC and R on circle cAB. Then the circles through $\{A,R,Q\},\{B,P,R\}$ and $\{C,Q,P\}$ have a point in common, called Miquel's point.

This theorem follows from the plane Miquel theorem by stereographic projection of the configuration on the sphere from D to a plane normal to OD, where O is the center of the sphere. The three circles through D correspond to straight lines in the plane forming the sides of a triangle. The other three circles correspond to those circles in the plane, which have the plane Miquel point in common and which corresponds to the spherical Miquel point.

4. Proof of Roberts' Theorem

Proof
Let the vertices of the tetrahedron be A,B,C,D. We intersect the spheres through A,B,D with face ABD and get a plane Miquel configuration with triangle ABD and the blue circles through $\{A,A',X\},\{B,X,B'\},\{D,B',A'\}$ intersecting in Miquel point R (Figure 4).
The same is done with the spheres through B,C,D and face BCD yielding the plane Miquel configuration with triangle BCD and the orange circles through $\{B,Y,B'\},\{C,C',Y\},\{D,B',C'\}$ intersecting in Miquel point P (Figure 5).
Finally the spheres through A,C,D are intersected with face ACD giving us the plane Miquel configuration with triangle ACD and the green circles through $\{A,Z,A'\},\{C,Z,C'\},\{D,A',C'\}$ intersecting in Miquel point Q (Figure 6).
Now we have three circles (blue, orange, green) on the sphere through D intersecting pairwise in points A',B',C'. Further we have point P on the orange circle through $\{B',C'\}$, point Q on the green circle through $\{A',C'\}$ and point R on the blue circle through $\{A',B'\}$. From the spherical Miquel theorem it follows that the circles through $\{A',R,Q\},\{B',P,R\}$ and $\{C',Q,P\}$ have a point M (on the sphere through D) in common (Figure 7).

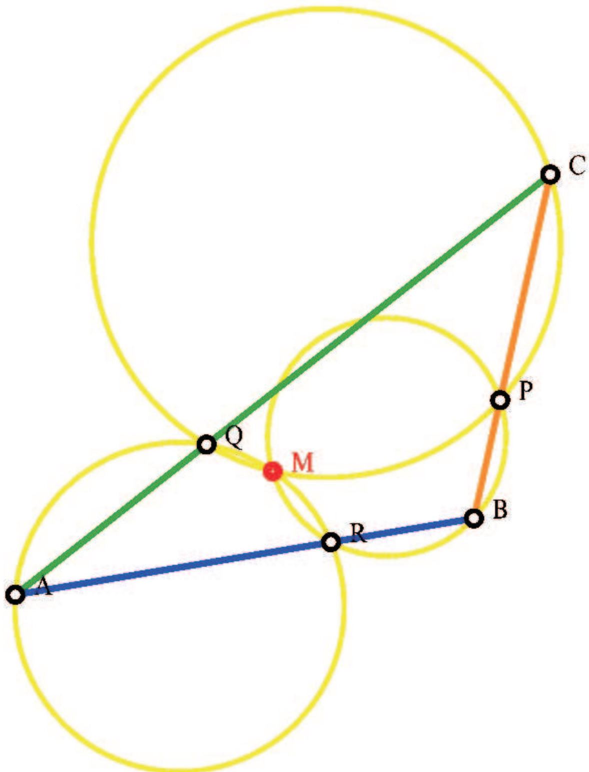

Figure 2: *Miquel's Triangle Theorem.*

The points A', R, Q lie on the sphere through A, and so this is also true for the circle through $\{A', R, Q\}$ and therefore for point M (Figure 8).
The points B', P, R lie on the sphere through B, and so does the circle through these points and particularly point M (Figure 9).
The points C', Q, P lie on the sphere through C, and again also the circle through these points and point M lie on this sphere (Figure 10).
So M is common to all four spheres! □

5. Conclusion

The proof shows us the possibility of using a theorem valid in the plane and on the sphere to establish an analogous fact in space. One might expect that if the $2d - 3d$ - correspondence of the elements (circles, spheres, etc.) is of a certain 'symmetric' kind the method can be used in other cases.

Roberts' theorem gives rise to the following question: For which 6-tuples of edgepoints the distance is constant between the intersection point of the four spheres and the circumcenter of the tetrahedron?

The analogous question arising from the Miquel theorem is easier and will be treated in a further paper. There is an interesting connection between certain hyperboloids and the Miquel points.

References

[NAC64] ALTSHILLER-COURT N.: *Modern Pure Solid Geometry.*. New York: Chelsea, 1964.

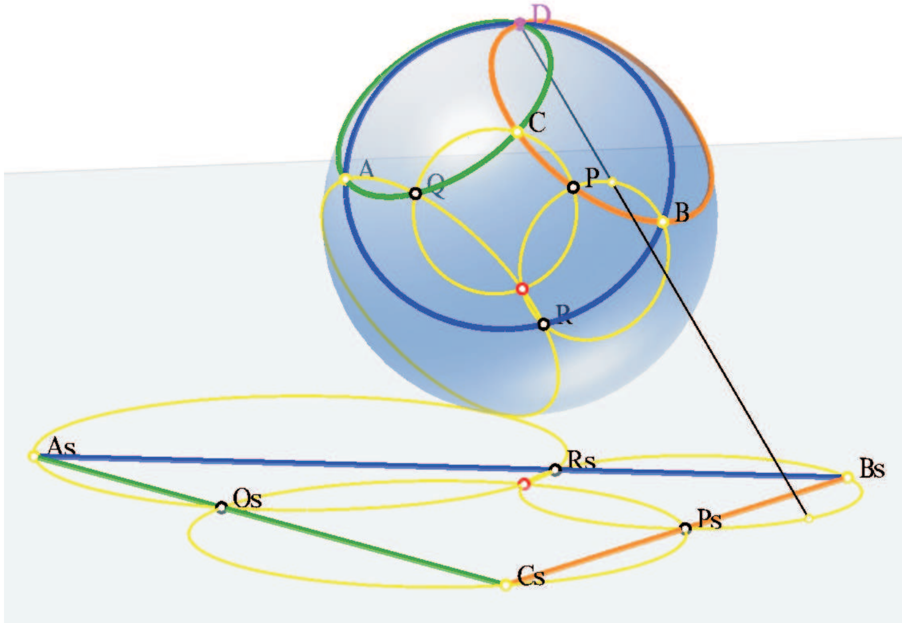

Figure 3: *Miquel's Triangle Theorem on the Sphere.*

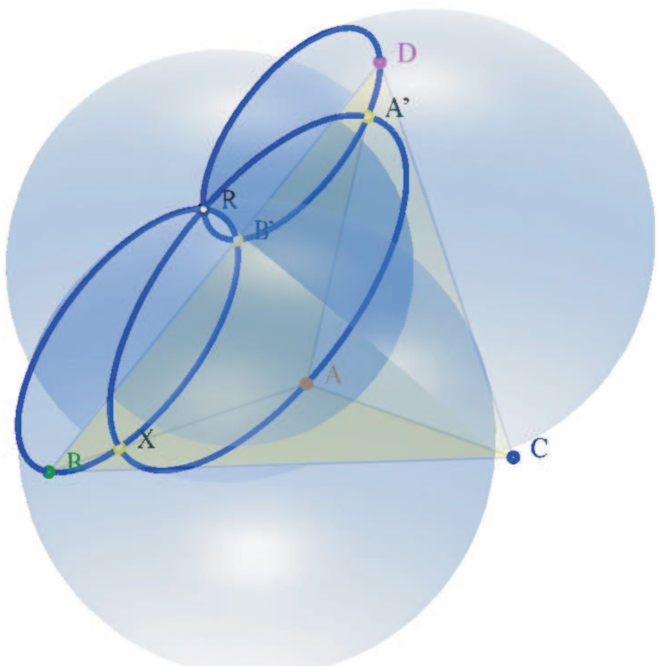

Figure 4: *Plane Miquel configuration with triangle ABD.*

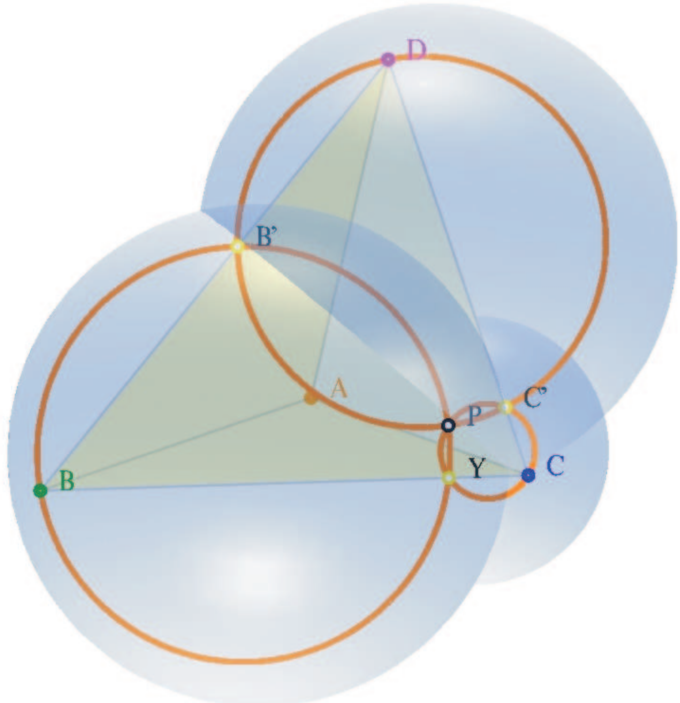

Figure 5: *Plane Miquel configuration with triangle BCD.*

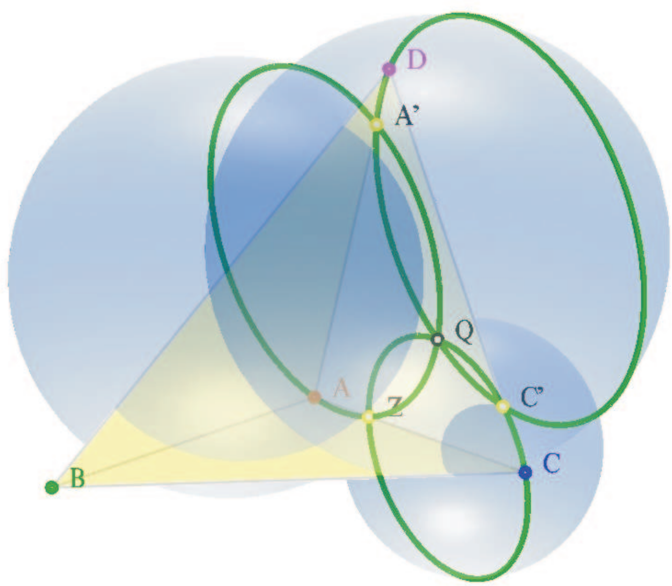

Figure 6: *Plane Miquel configuration with triangle ACD.*

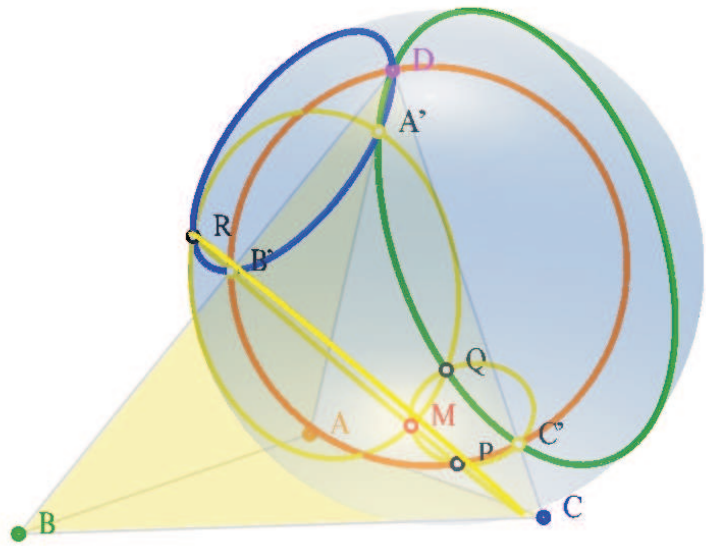

Figure 7: *Miquel configuration on the sphere through D.*

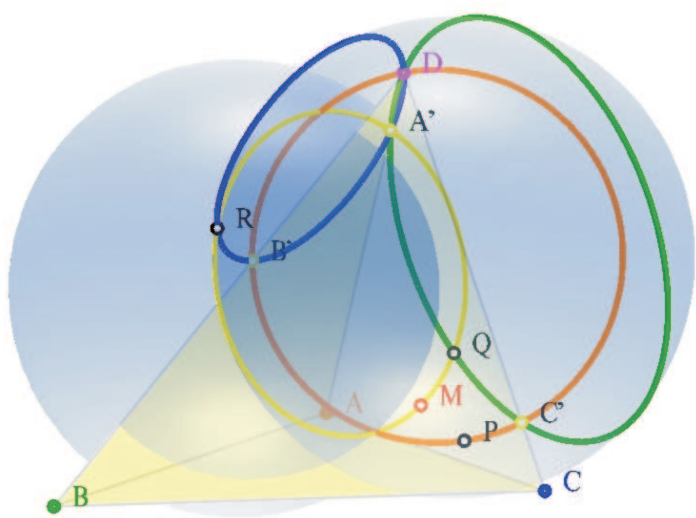

Figure 8: *M is on the sphere through A.*

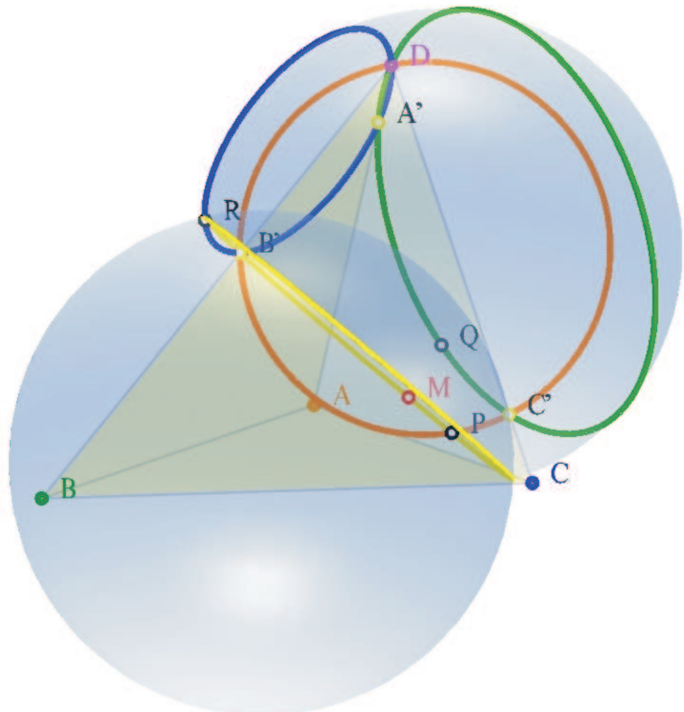

Figure 9: *M is on the sphere through B.*

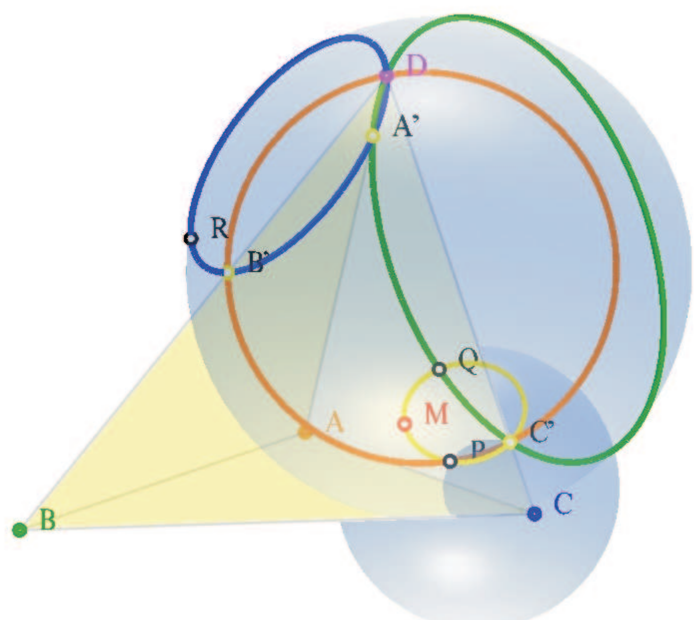

Figure 10: *M is on the sphere through C.*