Plane-Configurations of the Tetrahedron

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1. Gergonne's theorem

For a triangle we have the following fact ([RH95], pp.149,150):

The point of intersection of the internal or external bisector of a vertex with its opposite side divides this side in the ratio of the lengths of the sides concurring in that vertex.

The analog is also true for a tetrahedron ([NAC64], §235, p.80) (Figures 1, 2):

The point of intersection of the internal or external bisecting plane of an edge k with its opposite edge divides this edge in the ratio of the areas of the triangles sharing edge k.

Proof
For tetrahedron $ABCD$ we introduce some abbreviations:
$\overline{AB}$:= oriented length of segment AB,
$\overline{ABC}$:= area of triangle ABC,
$\overline{ABCD}$:= volume of the tetrahedron,
h_{AB} := height of triangle ABC on side AB,
H_{ABC} := height of the terahedron on plane ABC.
Now let X_i be the point of intersection of the internal bisecting plane of edge CD with its opposite edge and d the distance of X_i to planes ACD and BCD. Then we have

$$\overline{ACDX_i} = 1/3 * H_{ABC} * \overline{ACX_i} = 1/3 * d * \overline{ACD}$$
$$\overline{BCDX_i} = 1/3 * H_{ABC} * \overline{BCX_i} = 1/3 * d * \overline{BCD}$$

from which follows

$$\overline{ACD} : \overline{BCD} = \overline{ACX_i} : \overline{BCX_i} = (1/2 * \overline{AX_i} * h_{AB}) : (1/2 * \overline{X_iB} * h_{AB}) = \overline{AX_i} : \overline{X_iB}$$

The same is true for intersection point X_a of the external bisecting plane of edge CD with its opposite edge:

$$\overline{ACD} : \overline{BCD} = \overline{ACX_a} : \overline{BCX_a} = -(1/2 * \overline{AX_a} * h_{AB}) : (1/2 * \overline{X_aB} * h_{AB}) = -\overline{AX_a} : \overline{X_aB}$$

□

2. Menelaos' and Ceva's theorem

Again we denote by $\overline{XY}$ the oriented distance of points X and Y, i.e. $\overline{XY} = -\overline{YX}$. Menelaos' theorem is basic for the following considerations and must be mentioned therefore - however without proof (see [RH95], p.147):

For a triangle ABC and points P on side AB, Q on side BC, R on side CA the following is valid:

$$P, Q, R \text{ are collinear} \Leftrightarrow (\overline{AP} : \overline{PB}) * (\overline{BQ} : \overline{QC}) * (\overline{CR} : \overline{RA}) = -1$$

Later we also need the dual statement which is Ceva's theorem ([RH95], p.137):

For a triangle ABC and points P, Q, R on its sides as before the following holds:

$$Lines(A,Q),(B,R),(C,P)\ are\ concurrent \Leftrightarrow (\overline{AP}:\overline{PB})*(\overline{BQ}:\overline{QC})*(\overline{CR}:\overline{RA}) = 1$$

3. Four special lines of the triangle

From Menelaos' theorem a remarkable fact for triangles can be derived ([RH95], p.149):

Let be given an angle bisecting line in each vertex of a triangle. Then their points of intersection with opposite sides are collinear if either

1. all bisecting lines are external angle bisectors or
2. one bisecting line is external and the other two are internal angle bisectors.

So altogether we get four lines from four triples of collinear points (Figure 3).

4. Cesáro's theorem

Again with Menelaos' theorem and Gergonne's theorem we can show a similiar result for the tetrahedron ([NAC64], §237, p.81):

At three edges concurrent in one vertex let be given either

1. three external bisecting planes or
2. one external and two internal bisecting planes.

Then their points of intersection with opposite edges are collinear.

Proof
First we take the case of external bisecting planes at edges AD, BD, CD and their points of intersection X_{a1}, X_{a2}, X_{a3} with opposite edges respectively (Figure 4). With Gergonne's theorem (and same denotations) we have

$$\begin{aligned}
\overline{ABD}:\overline{BCD} &= -(\overline{AX_{a2}}:\overline{X_{a2}C})\\
\overline{ACD}:\overline{ABD} &= -(\overline{CX_{a1}}:\overline{X_{a1}B})\\
\overline{BCD}:\overline{ACD} &= -(\overline{BX_{a3}}:\overline{X_{a3}A})
\end{aligned}$$

This implies

$$(\overline{AX_{a2}}:\overline{X_{a2}C})*(\overline{CX_{a1}}:\overline{X_{a1}B})*(\overline{BX_{a3}}:\overline{X_{a3}A}) = -((\overline{ABD}:\overline{BCD})*(\overline{ACD}:\overline{ABD})*(\overline{BCD}:\overline{ACD})) = -1$$

and so points X_{a1}, X_{a2}, X_{a3} are collinear by Menelaos' theorem.
In the second case we have an external bisecting plane at edge AD and internal bisecting planes at edges BD and CD. The corresponding intersection points with opposite edges are X_a, X_{i1} and X_{i2} respectively (Figure 5). Again with Gergonne's theorem we have

$$\begin{aligned}
\overline{ABD}:\overline{BCD} &= \overline{AX_{i1}}:\overline{X_{i1}C}\\
\overline{BCD}:\overline{ACD} &= \overline{BX_{i2}}:\overline{X_{i2}A}\\
\overline{ACD}:\overline{ABD} &= -(\overline{CX_a}:\overline{X_aB})
\end{aligned}$$

and further

$$(\overline{AX_{i1}}:\overline{X_{i1}C})*(\overline{CX_a}:\overline{X_aB})*(\overline{BX_{i2}}:\overline{X_{i2}A}) = -((\overline{ABD}:\overline{BCD})*(\overline{ACD}:\overline{ABD})*(\overline{BCD}:\overline{ACD})) = -1$$

which means that points X_a, X_{i1}, X_{i2} are collinear. $\square$

5. Eight special planes of the tetrahedron

Now we are able to formulate the tetrahedron's analog to the four special lines of a triangle:

The six intersection points of all edges with bisecting planes of their opposite edges are coplanar if either

1. all bisecting planes are external or

2. bisecting planes at edges concurring in a vertex are external, the others internal or
3. bisecting planes at one opposite pair of edges are external, the others internal.

These three cases yield **one**, **four** and **three** planes repectively, altogether eight planes.

Proof
Let $X_1, X_2, X_3, X_4, X_5, X_6$ be the points of intersection of the bisecting planes with edges CD, BD, BC, AD, AC, AB respectively.

1. By Cesáro's theorem X_1, X_2, X_3 (vertex A, case 1) and X_1, X_4, X_5 (vertex B, case 1) are collinear. So X_1, X_2, X_3, X_4, X_5 are coplanar. But with Cesáro X_2, X_4, X_6 (vertex C, case 1) are also collinear and therefore point X_6 lies on the plane of the other five points.
2. Let the external bisecting planes be given at the edges concurring in vertex D, the internal bisecting planes at the remaining edges. With Cesáro X_3, X_5, X_6 (vertex D, case 1) and X_1, X_2, X_3 (vertex A, case 2) are collinear. So X_1, X_2, X_3, X_5, X_6 are coplanar. But X_2, X_4, X_6 (vertex C, case 2) are also collinear and so X_4 lies on the plane of the other five points.
3. Let the external bisecting planes be given at edges AD and BC. With Cesáro (case 2) applied to vertices D, A, and C the statement follows as in case 2.

□

6. Four-line-configurations of the triangle

Let be given a triangle ABC and some points P, Q, R (but no vertex) on sides AB, BC, CA respectively. Let P', Q', R' be the harmonic conjugates of P, Q, R on the corresponding triangle sides, i.e.

$$(A,B,P,P') = (B,C,Q,Q') = (C,A,R,R') = -1$$

Then

$$P,Q,R\ collinear \Leftrightarrow P,Q',R'\ collinear \Leftrightarrow P',Q,R'\ collinear \Leftrightarrow P',Q',R\ collinear \Leftrightarrow lines\ (A,Q'),(B,R'),(C,P')\ concurrent.$$

Proof
We shortly write XY instead of $\overline{XY}$ for the oriented length of line segment XY. $(A,B,P,P') = (B,C,Q,Q') = (C,A,R,R') = -1$ means that

$$\begin{aligned} \frac{PA}{PB} : \frac{P'A}{P'B} &= -1\ or\ \frac{AP}{PB} = -\frac{AP'}{P'B} \\ \frac{QB}{QC} : \frac{Q'B}{Q'C} &= -1\ or\ \frac{BQ}{QC} = -\frac{BQ'}{Q'C} \\ \frac{RC}{RA} : \frac{R'C}{R'A} &= -1\ or\ \frac{CR}{RA} = -\frac{CR'}{R'A} \end{aligned}$$

Now with Menelaos' theorem we have

$$P,Q,R\ collinear \Leftrightarrow \frac{AP}{PB}\frac{BQ}{QC}\frac{CR}{RA} = -1 \Leftrightarrow \frac{AP}{PB}\frac{BQ'}{Q'C}\frac{CR'}{R'A} = -1 \Leftrightarrow P,Q',R'\ collinear$$

and with Ceva's theorem

$$P,Q,R\ collinear \Leftrightarrow \frac{AP}{PB}\frac{BQ}{QC}\frac{CR}{RA} = -1 \Leftrightarrow \frac{AP'}{P'B}\frac{BQ'}{Q'C}\frac{CR'}{R'A} = 1 \Leftrightarrow lines\ (A,Q'),(B,R'),(C,P')\ concurrent$$

The remaining equivalences follow the same way. □

If we intersect the sides of a triangle ABC with a line yielding intersection points P, Q, R (unequal to any vertex) on sides AB, BC, CA respectively and construct the harmonic conjugate points P', Q', R' of P, Q, R, then the assumptions of the last theorem are met and we have a four-line-configuration from four collinear point triples $(P,Q,R), (P,Q',R'), (P',Q,R'), (P',Q',R)$.

If P, Q, R are intersection points of external and P', Q', R' of internal angle bisectors in vertices C, A, B with corresponding opposite sides, then we know that $(A,B,P,P') = (B,C,Q,Q') = (C,A,R,R') = -1$ (section 1) and P, Q, R are collinear (section 3). So this constitutes a special example of a four-line-configuration.

7. Eight-plane-configurations of the tetrahedron

Let be given a tetrahedron $ABCD$ and some points P,Q,R,S,T,U (but no vertex) on edges AB,BC,CA,AD,BD,CD respectively. Let P',Q',R',S',T',U' be the harmonic conjugates of P,Q,R,S,T,U on the corresponding edges, i.e.

$$(A,B,P,P') = (B,C,Q,Q') = (C,A,R,R') = (A,D,S,S') = (B,D,T,T') = (C,D,U,U') = -1$$

Then

$$P,Q,R,S,T,U \text{ coplanar} \Leftrightarrow$$

$$P,Q,R,S',T',U' \text{ coplanar} \Leftrightarrow P',Q,R',S',T,U \text{ coplanar} \Leftrightarrow P',Q',R,S,T',U \text{ coplanar} \Leftrightarrow P,Q',R',S,T,U' \text{ coplanar} \Leftrightarrow$$

$$P',Q,R',S,T',U' \text{ coplanar} \Leftrightarrow P',Q',R,S',T,U' \text{ coplanar} \Leftrightarrow P,Q',R',S',T',U \text{ coplanar} \Leftrightarrow$$

$$\text{planes } (A,D,Q'),(B,D,R'),(C,D,P'),(A,B,U'),(B,C,S'),(C,A,T') \text{ concurrent}.$$

Proof
Because each triple $(P,Q,R),(P,S,T),(Q,T,U),(R,S,U)$ lies in a plane of the tetrahedron, we have the equivalence

$$P,Q,R,S,T,U \text{ coplanar} \Leftrightarrow P,Q,R \wedge P,S,T \wedge Q,T,U \wedge R,S,U \text{ collinear } (*).$$

Applying our theorem from section 6 we get seven more equivalences:

$$(*) \Leftrightarrow P,Q,R \wedge P,S',T' \wedge Q,T',U' \wedge R,S',U' \text{ collinear} \Leftrightarrow P,Q,R,S',T',U' \text{ coplanar},$$

$$(*) \Leftrightarrow P',Q,R' \wedge P',S',T \wedge Q,T,U \wedge R',S',U \text{ collinear} \Leftrightarrow P',Q,R',S',T,U \text{ coplanar},$$

$$(*) \Leftrightarrow P',Q',R \wedge P',S,T' \wedge Q',T',U \wedge R,S,U \text{ collinear} \Leftrightarrow P',Q',R,S,T',U \text{ coplanar},$$

$$(*) \Leftrightarrow P,Q',R' \wedge P,S,T \wedge Q',T,U' \wedge R',S,U' \text{ collinear} \Leftrightarrow P,Q',R',S,T,U' \text{ coplanar},$$

$$(*) \Leftrightarrow P',Q,R' \wedge P',S,T' \wedge Q,T',U' \wedge R',S,U' \text{ collinear} \Leftrightarrow P',Q,R',S,T',U' \text{ coplanar},$$

$$(*) \Leftrightarrow P',Q',R \wedge P',S',T \wedge Q',T,U' \wedge R,S',U' \text{ collinear} \Leftrightarrow P',Q',R,S',T,U' \text{ coplanar},$$

$$(*) \Leftrightarrow P,Q',R' \wedge P,S',T' \wedge Q',T',U \wedge R',S',U \text{ collinear} \Leftrightarrow P,Q',R',S',T',U \text{ coplanar}.$$

To show the equivalence

$$P,Q,R,S,T,U \text{ coplanar} \Leftrightarrow \text{planes } (A,D,Q'),(B,D,R'),(C,D,P'),(A,B,U'),(B,C,S'),(C,A,T') \text{ concurrent}$$

let's first assume that P,Q,R,S,T,U are coplanar. Then P,Q,R are collinear which implies (theorem section 6) that planes $(A,D,Q'),(B,D,R'),(C,D,P')$ have a common point D_0 in plane (A,B,C), i.e. they are intersecting in line (D,D_0). In the same way planes $(A,B,U'),(A,C,T'),(A,D,Q')$ have a common point A_0 in plane (B,C,D) and therefore intersect in line (A,A_0). This implies that point $Z := (D,D_0)\cap(A,A_0)$ belongs to five of the six planes in question. Finally planes $(A,B,U'),(B,C,S'),(B,D,R')$ have a common point B_0 in plane (A,C,D) and are thus intersecting in line (B,B_0). Because Z is element of planes (A,B,U') and (B,D,R') it must belong to line (B,B_0) and consequently to plane (B,C,S') thus being a common point of all six planes.

To show the inverse implication let the six planes be concurrent in a point Z. This implies that line (D,Z) is common to planes $(A,D,Q'),(B,D,R'),(C,D,P')$. Its point of intersection with plane (A,B,C) is a common point of lines $(A,Q'),(B,R'),(C,P')$. Similarly the line triples $((A,T'),(B,S'),(D,P'))$, $((B,U'),(C,T'),(D,Q'))$ and $((A,U'),(C,S'),(D,R'))$ are concurrent. Applying theorem (section 6) we then have the collinearity of triples $(P,Q,R),(P,S,T),(Q,T,U),(R,S,U)$ which implies the coplanarity of points P,Q,R,S,T,U. □

If we intersect a tetrahedron $ABCD$ with a plane yielding intersection points P,Q,R,S,T,U (unequal to any vertex) on edges AB,BC,CA,AD,BD,CD respectively and construct the harmonic conjugate points P',Q',R',S',T',U' of P,Q,R,S,T,U, then the assumptions of the last theorem are met and we have an eight-plane-configuration from eight coplanar six-tuples $(P,Q,R,S,T,U),(P,Q,R,S',T',U'),(P',Q,R',S',T,U),(P',Q',R,S,T',U),(P,Q',R',S,T,U'),(P',Q,R',S,T',U')$, $(P',Q',R,S',T,U'),(P,Q',R',S',T',U)$.

If P,Q,R,S,T,U are intersection points of external and P',Q',R',S',T',U' of internal angle bisectors in edges CD,AD,BD,BC,AC,AB with corresponding opposite edges, then from Gergonne's theorem (section 1) we know that $(A,B,P,P') = (B,C,Q,Q') = (C,A,R,R') = (A,D,S,S') = (B,D,T,T') = (C,D,U,U') = -1$ and from section 5 that P,Q,R,S,T,U are coplanar. So this constitutes a special example of an eight-plane-configuration.

References

[RH95] HONSBERGER R.: *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, The Mathematical Association of America, 1995.

[NAC64] ALTSHILLER-COURT N.: *Modern Pure Solid Geometry*, New York: Chelsea, 1964.

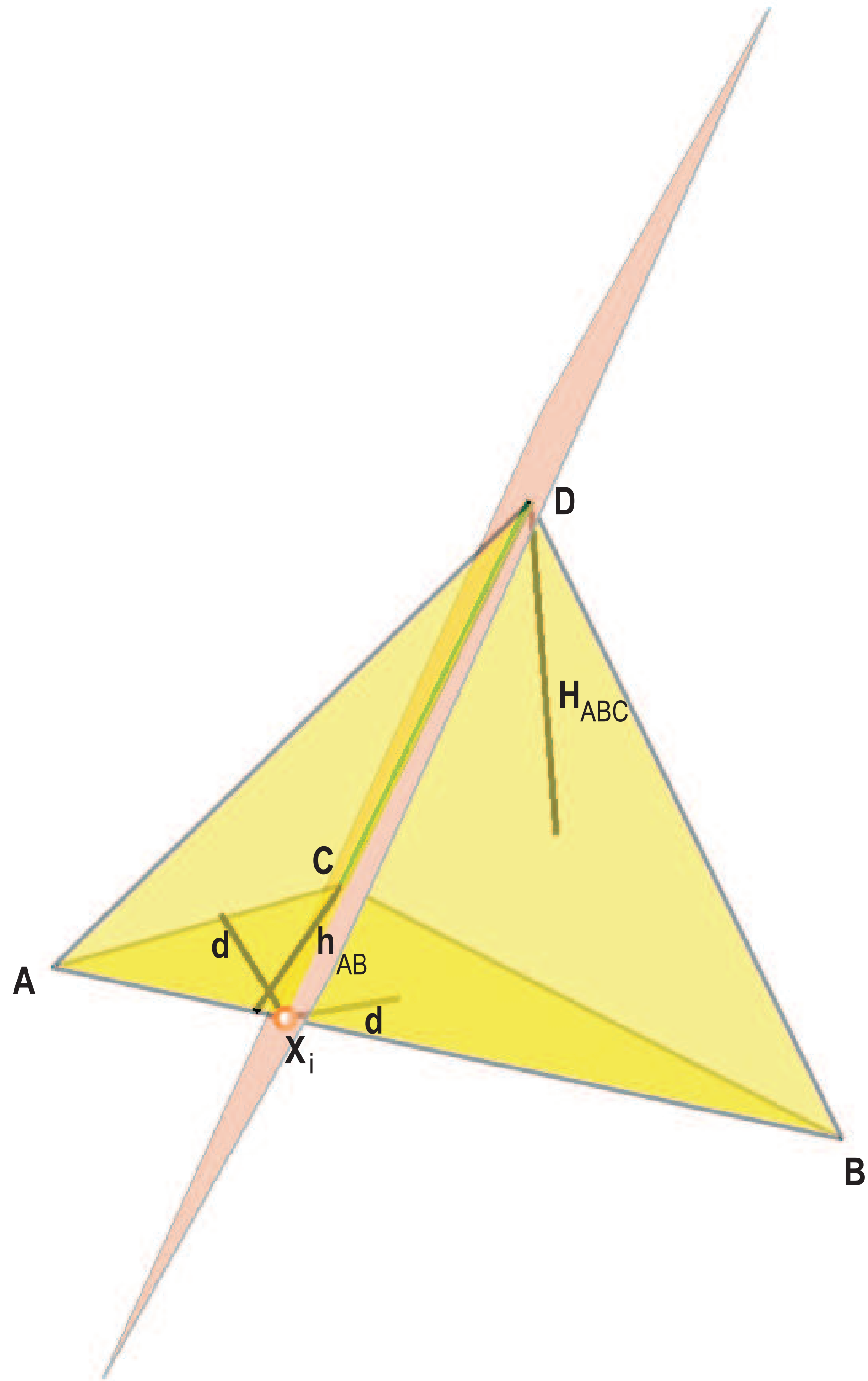

Figure 1: *Gergonne - internal angle bisector*

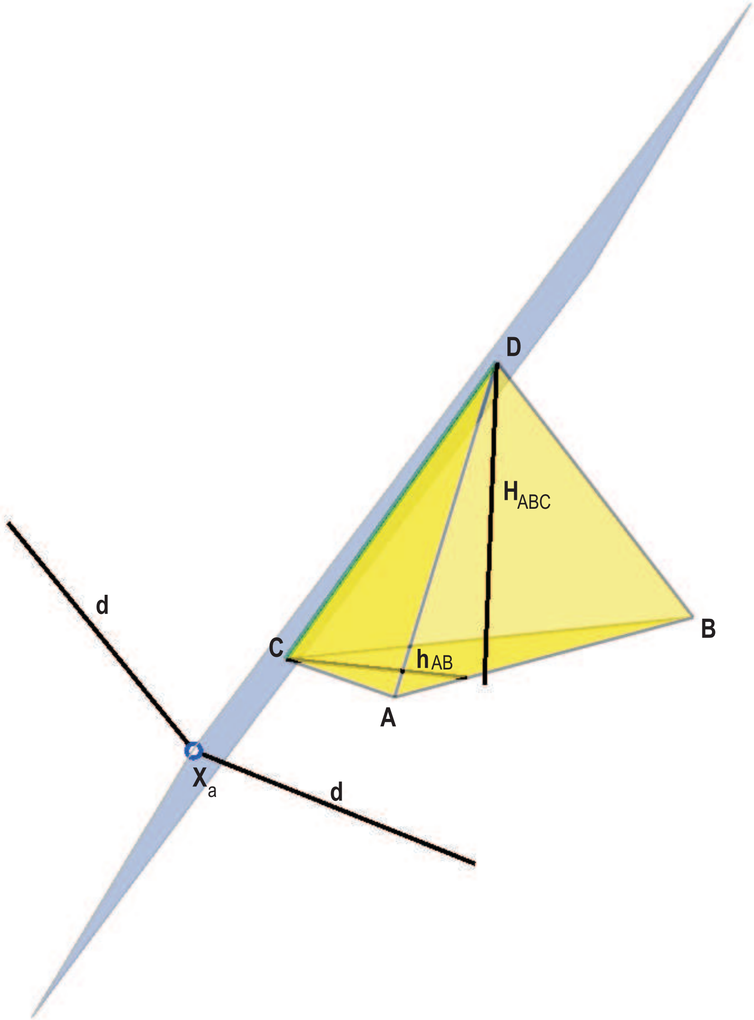

Figure 2: *Gergonne - external angle bisector*

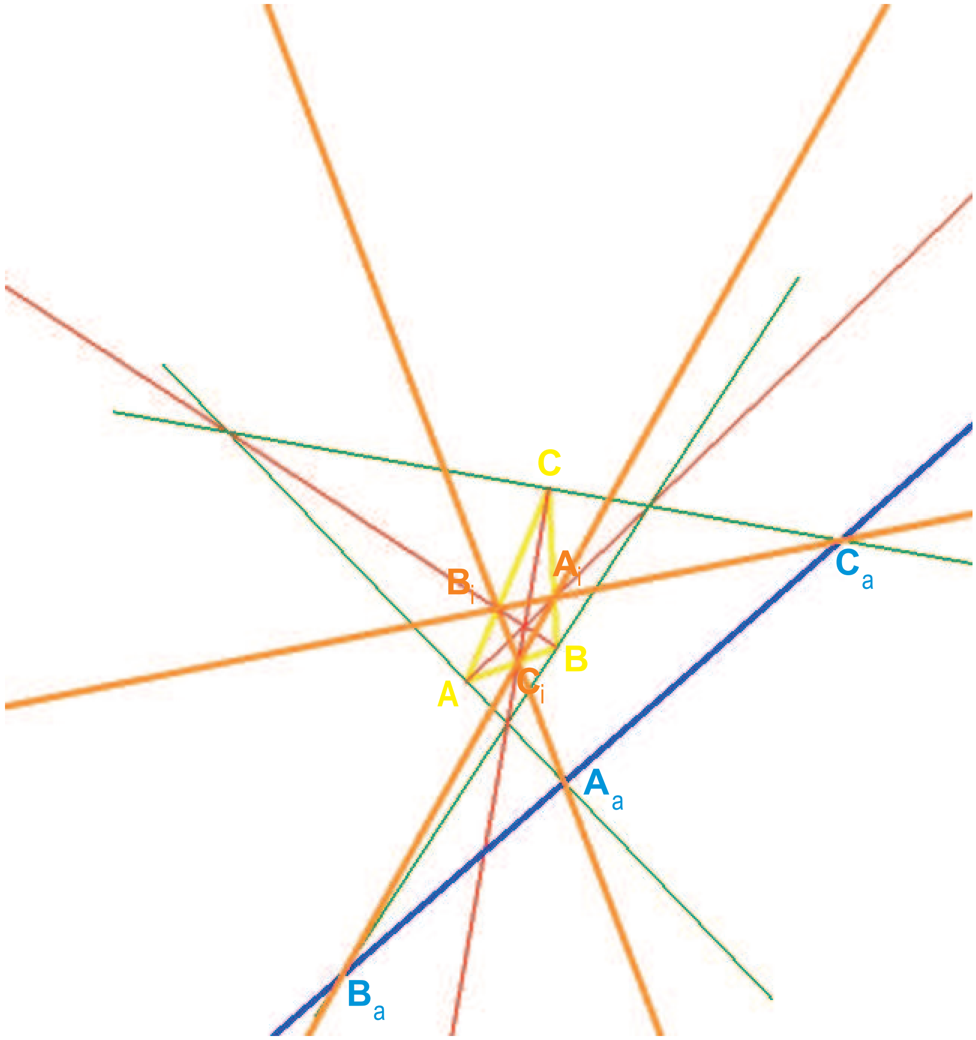

Figure 3: *Four special lines*

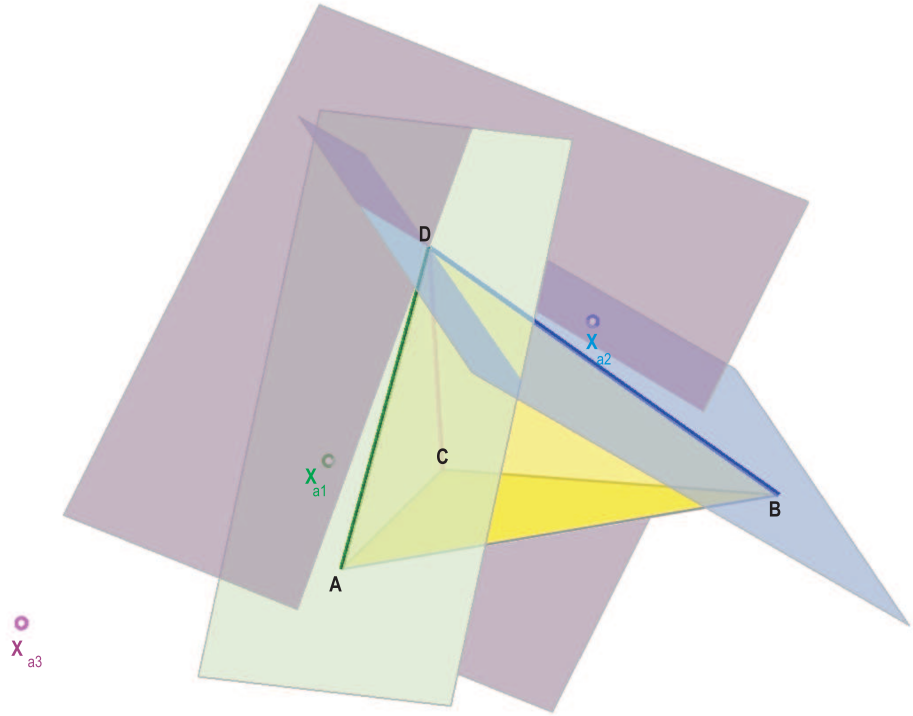

Figure 4: *Cesáro - three external angle bisectors*

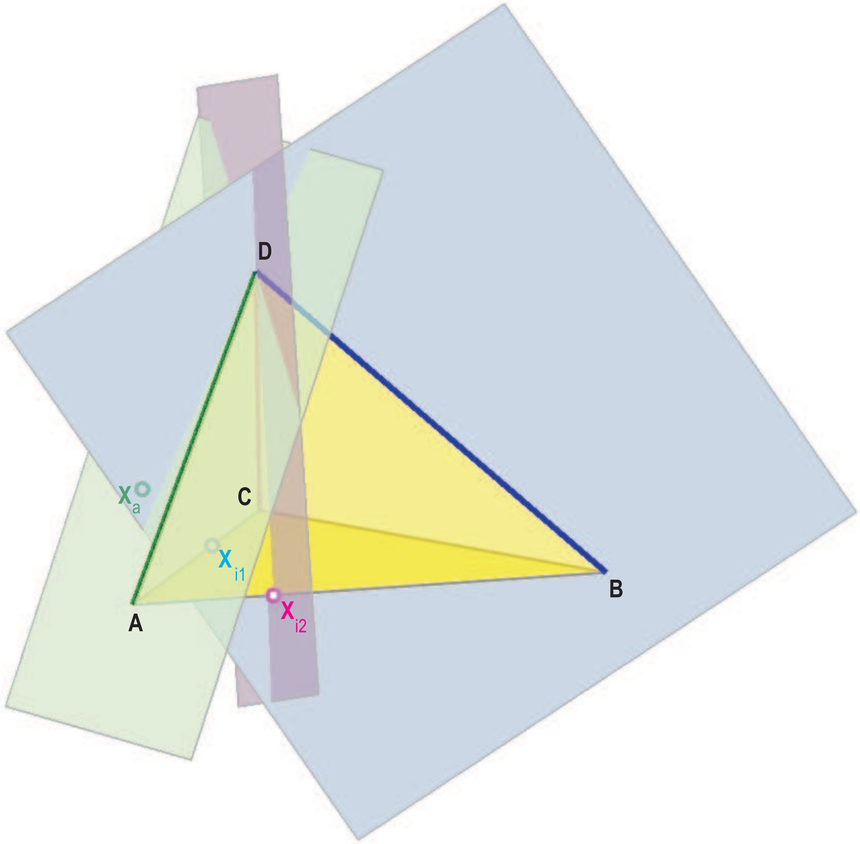

Figure 5: *Cesáro - one external, two internal angle bisectors*